

# Generalized $T$ - $Q$ relations and the open spin- $s$ XXZ chain with nondiagonal boundary terms

Rashad Baiyasi<sup>1</sup> and Rajan Murgan<sup>2</sup>

Department of Physics,  
Saginaw Valley State University,  
7400 Bay Road University Center, MI 48710 USA

## Abstract

We consider the open spin- $s$  XXZ quantum spin chain with nondiagonal boundary terms. By exploiting certain functional relations at roots of unity, we derive a generalized form of  $T$ - $Q$  relation involving more than one independent  $Q(u)$ , which we use to propose the Bethe-ansatz-type expressions for the eigenvalues of the transfer matrix. At most two of the boundary parameters are set to be arbitrary and the bulk anisotropy parameter has values  $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \dots$ . We also provide numerical evidence for the completeness of the Bethe-ansatz-type solutions derived, using  $s = 1$  case as an example.

---

<sup>1</sup>email:ribaiyas@svsu.edu

<sup>2</sup>e-mail: rmurgan@svsu.edu

# 1 Introduction

Considerable effort has been put into solving integrable quantum spin chains for many years. In particular, integrable open quantum spin chains have attracted much interest over the years. In this regard, open XXX and XXZ quantum spin chains have been extensively investigated due to their growing applications in fields of physics such as statistical mechanics, string theory and condensed matter physics. Much progress has been made on the topic up to this point. Numerous successes in the past [1]-[8] (also refer to [9]-[31] and references therein, for other related work on the subject) have motivated further investigations of these models. In addition, in a series of publication, Bethe ansatz solutions have been derived for open spin-1/2 XXZ quantum spin chain where the boundary parameters obey a certain constraint. Readers are urged to refer to [32]-[39] for related work on the subject. Two sets of Bethe ansatz equations are needed there to obtain all  $2^N$  eigenvalues, where  $N$  is the number of sites. A special case of the above solution was generalized to open XXZ quantum spin chain with alternating spins by Doikou [40] using the functional relation approach proposed by Nepomechie in [34] to solve the spin-1/2 case. In [41], related work was carried out using the method in [32]. In [42], the spin-1/2 XXZ Bethe ansatz solution (for boundary parameters obeying certain constraint) is generalized to the spin- $s$  case by utilizing an approach based on the  $Q$ -operator and the  $T$ - $Q$  equation [43] (see below), which was developed earlier for the spin-1/2 XXZ chain in [38] and subsequently applied to the spin-1/2 XYZ chain in [44]. Two sets of Bethe ansatz equations are also needed there to produce all  $(2s + 1)^N$  eigenvalues, where again  $N$  represents the number of sites. This was later followed by another work for spin- $s$  with such constraint removed, but limiting the Bethe ansatz solutions for cases with at most two arbitrary boundary parameters for some special values of the bulk anisotropy parameter [45], namely  $\eta = \frac{i\pi}{p+1}$ , with  $p$  being even integers [45].

In a number of works cited above, the well known Baxter  $T - Q$  relation [43], with the following schematic form

$$t(u)Q(u) = Q(v) + Q(w) \tag{1.1}$$

has provided a way to obtain the Bethe ansatz equations for the eigenvalues of the transfer matrix  $t(u)$ . In [46, 47], a generalization of this relation involving two  $Q(u)$  of the following form for the open spin-1/2 XXZ quantum spin chain was given:

$$\begin{aligned} t(u)Q_1(u) &= Q_2(v) + Q_2(w) \\ t(u)Q_2(u) &= Q_1(v') + Q_1(w') \end{aligned} \tag{1.2}$$

Motivated by this solution, in this paper, we obtain the corresponding solution for the open spin- $s$  XXZ quantum spin chain. In addition, our work is motivated by the fact that

such  $T - Q$  relations for an open spin- $s$  XXZ quantum spin chain are novel structures and therefore merit further studies and investigation. We remark that a more general form of  $T - Q$  relations were found in [48] involving multiple  $Q(u)$  functions. Moreover, the relation of  $s = 1$  case to the supersymmetric sine-Gordon (SSG) model [49]-[54], especially the boundary SSG model [55]-[59], has inspired us to consider the problem. We stress that these results hold for cases with at most two arbitrary boundary parameters at roots of unity, namely when the bulk anisotropy parameter has values  $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \dots$ . We follow similar approach as given in [34]-[37] and [46] that was used to solve the  $s = 1/2$  case, which is based on functional relations obeyed by transfer matrix at roots of unity. This yields Bethe-ansatz-type solutions which give the eigenvalues. Our numerical analysis for  $s = 1$  case for  $N = 2$  and  $p = 3, 5$  yields all the eigenvalues as given in Tables 1 and 2. As in [42], we rely on fusion [5], [60]-[66], the truncation of the fusion hierarchy at roots of unity [67]-[69] and the Bazhanov-Reshetikin [70] solution of the RSOS models.

The outline of the paper is as follows: In Sec. 2, we review the construction of the fused  $R$  [60]-[64], [71]-[75] and  $K^\mp$  [5], [65, 66] matrices from the corresponding spin-1/2 matrices. One can refer to [76, 77] for some original work on spin-1/2  $K^\mp$  matrices. We then review the construction of commuting transfer matrices from these fused matrices (using Sklyanin's work [4], which relies on Cherednik's previous results [80]), together with some of their properties. Fusion hierarchy and functional relations obeyed by transfer matrices are also reviewed. In Sec. 3, the generalized  $T - Q$  relations are given along with some arguments behind their structure. This is done by exploiting the reviewed functional relations obeyed by the transfer matrices. From this, we derive the Bethe-ansatz-type equations for cases with at most two arbitrary boundary parameters at roots of unity, e.g.  $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \dots$ . We then present numerical results in Sec. 4 to illustrate the completeness of our solution, using  $s = 1$  as an example. Here, the Bethe roots and energy eigenvalues derived from the Bethe-ansatz-type equations (for some values of  $p$  and  $N$ ) are given. We remark that these eigenvalues coincide with the ones obtained from direct diagonalization of the open spin-1 XXZ chain Hamiltonian. Finally, we conclude the paper with discussion of the results and potential future works in Sec. 5.

## 2 Commuting spin- $s$ transfer matrices and functional relations at roots of unity

In this section, in order to make the paper relatively self contained, we review some crucial concepts on the construction of commuting transfer matrices for  $N$ -site open spin- $s$  XXZ quantum spin chain. Materials reviewed here on fused  $R$ ,  $K^\mp$  and higher spin transfer

matrices are mainly reproduced from [42]. Like the commuting transfer matrix for  $s = 1/2$ , constructed in [4], which we denote (following notations adopted in [42]) by  $t^{(\frac{1}{2}, \frac{1}{2})}(u)$ , whose auxiliary space as well as each of its  $N$  quantum spaces are two-dimensional, one can construct a transfer matrix  $t^{(j,s)}(u)$  whose auxiliary space is spin- $j$   $((2j+1)$ -dimensional) and each of its  $N$  quantum spaces are spin- $s$   $((2s+1)$ -dimensional), for any  $j, s \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$  using the fused  $R$  [60]-[64], [71]-[75] and  $K^\mp$  [5], [65, 66] matrices. These matrices serve as building blocks in the construction of the commuting transfer matrices for higher spins. We list them below with some of their properties. The fused- $R$  matrices can be constructed as given below,

$$R_{\{a\}\{b\}}^{(j,s)}(u) = P_{\{a\}}^+ P_{\{b\}}^+ \prod_{k=1}^{2j} \prod_{l=1}^{2s} R_{a_k b_l}^{(\frac{1}{2}, \frac{1}{2})}(u + (k+l-j-s-1)\eta) P_{\{a\}}^+ P_{\{b\}}^+, \quad (2.1)$$

where  $\{a\} = \{a_1, \dots, a_{2j}\}$ ,  $\{b\} = \{b_1, \dots, b_{2s}\}$ , and  $P_{\{a\}}^+$  is the symmetric projector given by

$$P_{\{a\}}^+ = \frac{1}{(2j)!} \prod_{k=1}^{2j} \left( \sum_{l=1}^k \mathcal{P}_{a_l, a_k} \right), \quad (2.2)$$

$\mathcal{P}$  is the permutation operator, with  $\mathcal{P}_{a_k, a_k} \equiv 1$ ; similar definition also holds for  $P_{\{b\}}^+$ .  $R^{(\frac{1}{2}, \frac{1}{2})}(u)$  is given by

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) = \begin{pmatrix} \text{sh}(u + \eta) & 0 & 0 & 0 \\ 0 & \text{sh } u & \text{sh } \eta & 0 \\ 0 & \text{sh } \eta & \text{sh } u & 0 \\ 0 & 0 & 0 & \text{sh}(u + \eta) \end{pmatrix}, \quad (2.3)$$

where  $\eta$  is the bulk anisotropy parameter. The fused  $R$  matrices satisfy the Yang-Baxter equations [78, 79]

$$R_{\{a\}\{b\}}^{(j,k)}(u-v) R_{\{a\}\{c\}}^{(j,s)}(u) R_{\{b\}\{c\}}^{(k,s)}(v) = R_{\{b\}\{c\}}^{(k,s)}(v) R_{\{a\}\{c\}}^{(j,s)}(u) R_{\{a\}\{b\}}^{(j,k)}(u-v). \quad (2.4)$$

The construction of the fused  $K^-$  matrices now readily follows [5], [65, 66],

$$\begin{aligned} K_{\{a\}}^{-(j)}(u) &= P_{\{a\}}^+ \prod_{k=1}^{2j} \left\{ \left[ \prod_{l=1}^{k-1} R_{a_l a_k}^{(\frac{1}{2}, \frac{1}{2})}(2u + (k+l-2j-1)\eta) \right] \right. \\ &\quad \times \left. K_{a_k}^{-(\frac{1}{2})}(u + (k-j-\frac{1}{2})\eta) \right\} P_{\{a\}}^+, \end{aligned} \quad (2.5)$$

where  $K^{-(\frac{1}{2})}(u)$  is the  $2 \times 2$  matrix whose components are given by [76, 77]

$$\begin{aligned} K_{11}^-(u) &= 2(\text{sh } \alpha_- \text{ch } \beta_- \text{ch } u + \text{ch } \alpha_- \text{sh } \beta_- \text{sh } u) \\ K_{22}^-(u) &= 2(\text{sh } \alpha_- \text{ch } \beta_- \text{ch } u - \text{ch } \alpha_- \text{sh } \beta_- \text{sh } u) \\ K_{12}^-(u) &= e^{\theta_-} \text{sh } 2u, \quad K_{21}^-(u) = e^{-\theta_-} \text{sh } 2u, \end{aligned} \quad (2.6)$$

where  $\alpha_- , \beta_- , \theta_-$  are the boundary parameters. The fused  $K^-$  matrices satisfy the boundary Yang-Baxter equations [80]

$$\begin{aligned} R_{\{a\}\{b\}}^{(j,s)}(u-v) K_{\{a\}}^{-(j)}(u) R_{\{a\}\{b\}}^{(j,s)}(u+v) K_{\{b\}}^{-(j)}(v) \\ = K_{\{b\}}^{-(j)}(v) R_{\{a\}\{b\}}^{(j,s)}(u+v) K_{\{a\}}^{-(j)}(u) R_{\{a\}\{b\}}^{(j,s)}(u-v). \end{aligned} \quad (2.7)$$

In addition, the fused  $K^+$  matrices are given by

$$K_{\{a\}}^{+(j)}(u) = \frac{1}{f^{(j)}(u)} K_{\{a\}}^{-(j)}(-u-\eta) \Big|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)}, \quad (2.8)$$

with the following normalization factor,

$$f^{(j)}(u) = \prod_{l=1}^{2j-1} \prod_{k=1}^l [-\xi(2u + (l+k+1-2j)\eta)]. \quad (2.9)$$

From the fused matrices, one constructs the higher spin transfer matrix  $t^{(j,s)}(u)$ ,

$$t^{(j,s)}(u) = \text{tr}_{\{a\}} K_{\{a\}}^{+(j)}(u) T_{\{a\}}^{(j,s)}(u) K_{\{a\}}^{-(j)}(u) \hat{T}_{\{a\}}^{(j,s)}(u). \quad (2.10)$$

The monodromy matrices are given by products of  $N$  fused  $R$  matrices,

$$\begin{aligned} T_{\{a\}}^{(j,s)}(u) &= R_{\{a\}, \{b^{[N]}\}}^{(j,s)}(u) \dots R_{\{a\}, \{b^{[1]}\}}^{(j,s)}(u), \\ \hat{T}_{\{a\}}^{(j,s)}(u) &= R_{\{a\}, \{b^{[1]}\}}^{(j,s)}(u) \dots R_{\{a\}, \{b^{[N]}\}}^{(j,s)}(u). \end{aligned} \quad (2.11)$$

These transfer matrices commute for different values of spectral parameter for any  $j, j' \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$  and any  $s \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ ,

$$[t^{(j,s)}(u), t^{(j',s)}(u')] = 0. \quad (2.12)$$

Furthermore, they also obey the fusion hierarchy [5, 42, 65, 66]<sup>1</sup>

$$t^{(j-\frac{1}{2},s)}(u-j\eta) t^{(\frac{1}{2},s)}(u) = t^{(j,s)}(u-(j-\frac{1}{2})\eta) + \delta^{(s)}(u-\eta) t^{(j-1,s)}(u-(j+\frac{1}{2})\eta), \quad (2.13)$$

$j = 1, \frac{3}{2}, \dots$ , where  $t^{(0,s)} = 1$ , and  $\delta^{(s)}(u)$  is given by

$$\delta^{(s)}(u) = \delta_0^{(s)}(u) \delta_1^{(s)}(u), \quad (2.14)$$

where

$$\begin{aligned} \delta_0^{(s)}(u) &= \left[ \prod_{k=0}^{2s-1} \xi(u + (s-k+\frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u) \text{sh}(2u+4\eta)}{\text{sh}(2u+\eta) \text{sh}(2u+3\eta)} \\ \delta_1^{(s)}(u) &= 2^4 \text{sh}(u+\alpha_-+\eta) \text{sh}(u-\alpha_-+\eta) \text{ch}(u+\beta_-+\eta) \text{ch}(u-\beta_-+\eta) \\ &\quad \times \text{sh}(u+\alpha_++\eta) \text{sh}(u-\alpha_++\eta) \text{ch}(u+\beta_++\eta) \text{ch}(u-\beta_++\eta). \end{aligned} \quad (2.15)$$

---

<sup>1</sup>See the appendix in [42] for more details on the fusion hierarchy.

To avoid confusion, we emphasize that the  $\delta^{(s)}(u)$  in [42] differs from the one given here by a shift in  $\eta$ .

Next, we list a few important properties of the rescaled “fundamental” transfer matrix  $\tilde{t}^{(\frac{1}{2},s)}(u)$  (defined below), which are useful in determining its eigenvalues. Following the definition of  $\tilde{t}^{(\frac{1}{2},s)}(u)$  as in [42], we have

$$\tilde{t}^{(\frac{1}{2},s)}(u) = \frac{1}{g^{(\frac{1}{2},s)}(u)^{2N}} t^{(\frac{1}{2},s)}(u), \quad (2.16)$$

where

$$g^{(\frac{1}{2},s)}(u) = \prod_{k=1}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta). \quad (2.17)$$

This transfer matrix has the following useful properties:

$$\tilde{t}^{(\frac{1}{2},s)}(u + i\pi) = \tilde{t}^{(\frac{1}{2},s)}(u) \quad (i\pi - \text{periodicity}) \quad (2.18)$$

$$\tilde{t}^{(\frac{1}{2},s)}(-u - \eta) = \tilde{t}^{(\frac{1}{2},s)}(u) \quad (\text{crossing}) \quad (2.19)$$

$$\tilde{t}^{(\frac{1}{2},s)}(0) = -2^3 \text{sh}^{2N}((s + \frac{1}{2})\eta) \text{ch } \eta \text{sh } \alpha_- \text{ch } \beta_- \text{sh } \alpha_+ \text{ch } \beta_+ \mathbb{I} \quad (\text{initial condition}) \quad (2.20)$$

$$\begin{aligned} \tilde{t}^{(\frac{1}{2},s)}(u) \Big|_{\eta=0} &= 2^3 \text{sh}^{2N} u \left[ -\text{sh } \alpha_- \text{ch } \beta_- \text{sh } \alpha_+ \text{ch } \beta_+ \text{ch}^2 u \right. \\ &\quad + \text{ch } \alpha_- \text{sh } \beta_- \text{ch } \alpha_+ \text{sh } \beta_+ \text{sh}^2 u \\ &\quad \left. - \text{ch}(\theta_- - \theta_+) \text{sh}^2 u \text{ch}^2 u \right] \mathbb{I} \quad (\text{semi-classical limit}) \end{aligned} \quad (2.21)$$

where  $\mathbb{I}$  is the identity matrix.

Due to the commutativity property (2.12), the corresponding simultaneous eigenvectors are independent of the spectral parameter. Hence, (2.18) - (2.21) hold for the corresponding eigenvalues as well. In addition to the above mentioned properties, for bulk anisotropy parameter values  $\eta = \frac{i\pi}{p+1}$ , with  $p = 1, 2, \dots$ , the “fundamental” transfer matrix,  $t^{(\frac{1}{2},s)}(u)$  (and hence each of the corresponding eigenvalues,  $\Lambda^{(\frac{1}{2},s)}(u)$ ) obeys functional relations of order  $p+1$  [34]-[36],

$$\begin{aligned} &t^{(\frac{1}{2},s)}(u) t^{(\frac{1}{2},s)}(u + \eta) \dots t^{(\frac{1}{2},s)}(u + p\eta) \\ &- \delta^{(s)}(u - \eta) t^{(\frac{1}{2},s)}(u + \eta) t^{(\frac{1}{2},s)}(u + 2\eta) \dots t^{(\frac{1}{2},s)}(u + (p-1)\eta) \\ &- \delta^{(s)}(u) t^{(\frac{1}{2},s)}(u + 2\eta) t^{(\frac{1}{2},s)}(u + 3\eta) \dots t^{(\frac{1}{2},s)}(u + p\eta) \end{aligned}$$

$$\begin{aligned}
& - \delta^{(s)}(u + \eta) t^{(\frac{1}{2}, s)}(u) t^{(\frac{1}{2}, s)}(u + 3\eta) t^{(\frac{1}{2}, s)}(u + 4\eta) \dots t^{(\frac{1}{2}, s)}(u + p\eta) \\
& - \delta^{(s)}(u + 2\eta) t^{(\frac{1}{2}, s)}(u) t^{(\frac{1}{2}, s)}(u + \eta) t^{(\frac{1}{2}, s)}(u + 4\eta) \dots t^{(\frac{1}{2}, s)}(u + p\eta) - \dots \\
& - \delta^{(s)}(u + (p-1)\eta) t^{(\frac{1}{2}, s)}(u) t^{(\frac{1}{2}, s)}(u + \eta) \dots t^{(\frac{1}{2}, s)}(u + (p-2)\eta) \\
& + \dots = f(u).
\end{aligned} \tag{2.22}$$

The scalar function  $f(u)$  (which can be expressed as  $f(u) = f_0(u)f_1(u)$ ) is given in terms of the boundary parameters  $\alpha_{\mp}, \beta_{\mp}, \theta_{\mp}$  (for odd  $p$ ) by

$$f_0(u) = \begin{cases} (-1)^{N+1} 2^{-4spN} \operatorname{sh}^{4sN}((p+1)u) \operatorname{th}^2((p+1)u), \\ \quad s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \\ (-1)^{N+1} 2^{-4spN} \operatorname{ch}^{4sN}((p+1)u) \operatorname{th}^2((p+1)u), \\ \quad s = 1, 2, 3, \dots \end{cases} \tag{2.23}$$

and

$$\begin{aligned}
f_1(u) = & -2^{3-2p} \Big( \\
& \operatorname{ch}((p+1)\alpha_-) \operatorname{ch}((p+1)\beta_-) \operatorname{ch}((p+1)\alpha_+) \operatorname{ch}((p+1)\beta_+) \operatorname{sh}^2((p+1)u) \\
& - \operatorname{sh}((p+1)\alpha_-) \operatorname{sh}((p+1)\beta_-) \operatorname{sh}((p+1)\alpha_+) \operatorname{sh}((p+1)\beta_+) \operatorname{ch}^2((p+1)u) \\
& + (-1)^N \operatorname{ch}((p+1)(\theta_- - \theta_+)) \operatorname{sh}^2((p+1)u) \operatorname{ch}^2((p+1)u) \Big),
\end{aligned} \tag{2.24}$$

for  $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$  and

$$\begin{aligned}
f_1(u) = & (-1)^{N+1} 2^{3-2p} \Big( \\
& \operatorname{ch}((p+1)\alpha_-) \operatorname{ch}((p+1)\beta_-) \operatorname{ch}((p+1)\alpha_+) \operatorname{ch}((p+1)\beta_+) \operatorname{sh}^2((p+1)u) \\
& - \operatorname{sh}((p+1)\alpha_-) \operatorname{sh}((p+1)\beta_-) \operatorname{sh}((p+1)\alpha_+) \operatorname{sh}((p+1)\beta_+) \operatorname{ch}^2((p+1)u) \\
& + \operatorname{ch}((p+1)(\theta_- - \theta_+)) \operatorname{sh}^2((p+1)u) \operatorname{ch}^2((p+1)u) \Big),
\end{aligned} \tag{2.25}$$

for  $s = 1, 2, 3, \dots$ . Note that  $f(u)$  satisfies

$$f(u + \eta) = f(u), \quad f(-u) = f(u), \tag{2.26}$$

and

$$f_0(u)^2 = \prod_{j=0}^p \delta_0^{(s)}(u + j\eta), \tag{2.27}$$

where  $\delta_0^{(s)}(u)$  is given by (2.15).

### 3 Generalized $T$ - $Q$ relations and Bethe ansatz

In this section, we give the main results of this paper. We derive the generalized  $T - Q$  relations for the transfer matrix eigenvalues and obtain the Bethe-ansatz-type equations, for cases where at most two of the boundary parameters  $\alpha_{\pm}$  or  $\beta_{\pm}$  are arbitrary, by adopting the steps given in [46] while setting  $\theta_- = \theta_+ = \theta$ , where  $\theta$  is also arbitrary. More on this is given below.

#### 3.1 $T - Q$ relations

The transfer matrix  $t^{(\frac{1}{2},s)}(u)$  and its eigenvalues  $(\Lambda^{(\frac{1}{2},s)}(u))$  obey the functional relations (2.22). We exploit this fact to obtain the  $T - Q$  relations. Following [70], one could recast the functional relations as the condition that the determinant of a certain matrix vanishes, namely

$$\det \mathcal{M}(u) = 0, \quad (3.1)$$

where  $\mathcal{M}(u)$  is given by the  $(p+1) \times (p+1)$  matrix

$$\mathcal{M}(u) = \begin{pmatrix} \Lambda^{(\frac{1}{2},s)}(u) & -\frac{\delta^{(s)}(u)}{h^{(1)}(u)} & 0 & \dots & 0 & -\frac{\delta^{(s)}(u-\eta)}{h^{(2)}(u-\eta)} \\ -h^{(1)}(u) & \Lambda^{(\frac{1}{2},s)}(u+\eta) & -h^{(2)}(u+\eta) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h^{(2)}(u-\eta) & 0 & 0 & \dots & -h^{(1)}(u+(p-1)\eta) & \Lambda^{(\frac{1}{2},s)}(u+p\eta) \end{pmatrix} \quad (3.2)$$

where  $h^{(1)}(u)$  and  $h^{(2)}(u)$  are functions which are  $i\pi$ -periodic, but otherwise not yet specified. We note that the above matrix has the following symmetry,

$$T \mathcal{M}(u) T^{-1} = \mathcal{M}(u+2\eta), \quad T \equiv S^2, \quad (3.3)$$

where  $S$  is given by,

$$S = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (3.4)$$

Assuming that

$$\det \mathcal{M}(u) = 0, \quad (3.5)$$



then  $\mathcal{M}(u)$  has a null eigenvector  $v(u)$ ,

$$\mathcal{M}(u) v(u) = 0. \quad (3.6)$$

The symmetry (3.3) is consistent with

$$T v(u) = v(u + 2\eta), \quad (3.7)$$

which implies that  $v(u)$  has the form

$$v(u) = (Q_1(u), Q_2(u + \eta), \dots, Q_1(u - 2\eta), Q_2(u - \eta)), \quad (3.8)$$

with

$$Q_1(u) = Q_1(u + i\pi), \quad Q_2(u) = Q_2(u + i\pi). \quad (3.9)$$

That is, the components of  $v(u)$  are determined by *two* independent functions,  $Q_1(u)$  and  $Q_2(u)$ .<sup>2</sup>

The null eigenvector condition (3.6) together with the explicit forms of  $\mathcal{M}(u)$  and  $v(u)$ , given by (3.2) and (3.8) respectively, now lead to the following  $T - Q$  relations,

$$\Lambda^{(\frac{1}{2}, s)}(u) = \frac{\delta^{(s)}(u)}{h^{(1)}(u)} \frac{Q_2(u + \eta)}{Q_1(u)} + \frac{\delta^{(s)}(u - \eta)}{h^{(2)}(u - \eta)} \frac{Q_2(u - \eta)}{Q_1(u)}, \quad (3.10)$$

$$= h^{(1)}(u - \eta) \frac{Q_1(u - \eta)}{Q_2(u)} + h^{(2)}(u) \frac{Q_1(u + \eta)}{Q_2(u)}. \quad (3.11)$$

Due to the crossing symmetry (2.19) and

$$\delta^{(s)}(u) = \delta^{(s)}(-u - 2\eta), \quad (3.12)$$

which is the crossing property for  $\delta^{(s)}(u)$ , it is then natural to have the two terms in (3.10) transform into each other under crossing. Hence, we set

$$h^{(2)}(u) = h^{(1)}(-u - 2\eta), \quad (3.13)$$

and we make the following ansatz

$$Q_j(u) = \prod_{k=1}^{M_j} \sinh(u - u_k^{(j)}) \sinh(u + u_k^{(j)} + \eta), \quad (3.14)$$

which is consistent with the required periodicity (3.9) and crossing properties

$$Q_j(u) = Q_j(-u - \eta), \quad (3.15)$$

---

<sup>2</sup>In [45], all of the matrices  $\mathcal{M}(u)$  possess a stronger symmetry,  $S \mathcal{M}(u) S^{-1} = \mathcal{M}(u + \eta)$ , implying the null eigenvector with single  $Q(u)$ . We refer the reader to Sec. 3 of [46] for more detail discussion on this.

where  $j = 1, 2$ . In (3.14),  $\{u_k^{(j)}\}$  represents the Bethe roots (or zeros of  $Q_j(u)$ ) and there are  $M_j$  of these roots. Further, one can verify that the condition  $\det \mathcal{M}(u) = 0$  indeed implies the functional relations (2.22), if  $w(u)$  satisfies

$$f(u) = w(u) \prod_{j=0,2,\dots}^{p-1} \delta^{(s)}(u + j\eta) + \frac{1}{w(u)} \prod_{j=1,3,\dots}^p \delta^{(s)}(u + j\eta), \quad (3.16)$$

where

$$w(u) \equiv \frac{\prod_{j=1,3,\dots}^p h^{(2)}(u + j\eta)}{\prod_{j=0,2,\dots}^{p-1} h^{(1)}(u + j\eta)}. \quad (3.17)$$

It follows from (3.16) that the process of finding  $w(u)$  reduces to solving a quadratic equation, which when used together with (3.13) and (3.17), yields the explicit form for the function  $h^{(1)}(u)$ . Here, we consider even number of sites,  $N$ . Below, we give the solutions of (3.17) for  $h^{(1)}(u)$  for two cases:

I.  $\beta_-$  and  $\beta_+$  arbitrary while setting  $\alpha_{\pm} = 0$ ,  $\theta_- = \theta_+ = \theta = \text{arbitrary}$ .

$$h^{(1)}(u) = 4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{3}{2})\eta) \right]^{2N} \frac{\text{sh}^2(u + \eta) \text{sh}(2u + 4\eta)}{\text{sh}(2u + 3\eta)},$$

$$M_1 = sN + \frac{1}{2}(p + 1), \quad M_2 = M_1 - 1. \quad (3.18)$$

II.  $\alpha_-$  and  $\alpha_+$  arbitrary while setting  $\beta_{\pm} = 0$ ,  $\theta_- = \theta_+ = \theta = \text{arbitrary}$ .

$$h^{(1)}(u) = 4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{3}{2})\eta) \right]^{2N} \frac{\text{ch}^2(u + \eta) \text{sh}(2u + 4\eta)}{\text{sh}(2u + 3\eta)},$$

$$M_1 = sN + \frac{1}{2}(p + 1), \quad M_2 = M_1 - 1. \quad (3.19)$$

Now, using the analyticity of  $\Lambda^{(\frac{1}{2}, s)}(u)$ , given by (3.10) and (3.11), one can write down the Bethe-ansatz-type equations for the zeros  $\{u_j^{(1)}, u_j^{(2)}\}$  of  $Q_1(u)$  and  $Q_2(u)$ ,

$$\frac{\delta^{(s)}(u_j^{(1)}) h^{(2)}(u_j^{(1)} - \eta)}{\delta^{(s)}(u_j^{(1)} - \eta) h^{(1)}(u_j^{(1)})} = -\frac{Q_2(u_j^{(1)} - \eta)}{Q_2(u_j^{(1)} + \eta)}, \quad j = 1, 2, \dots, M_1, \quad (3.20)$$

$$\frac{h^{(1)}(u_j^{(2)} - \eta)}{h^{(2)}(u_j^{(2)})} = -\frac{Q_1(u_j^{(2)} + \eta)}{Q_1(u_j^{(2)} - \eta)}, \quad j = 1, 2, \dots, M_2. \quad (3.21)$$

We remark here that for each case, there are more than one solutions for  $h^{(1)}(u)$  that correspond to the above expression for  $w(u)$ . The solutions found are largely by trial and error, verifying numerically for small values of  $N$  that the eigenvalues can indeed be expressed as (3.10), (3.11) with  $Q(u)$ 's of the form (3.14).

To summarize, we have proposed that for the case where the bulk anisotropy parameter,  $\eta = \frac{i\pi}{p+1}$  with  $p$  being odd integers and that at most two of the boundary parameters are arbitrary, the eigenvalues  $\Lambda^{(\frac{1}{2},s)}(u)$  of the transfer matrix  $t^{(\frac{1}{2},s)}(u)$  for two cases (I and II) are given by a generalized form of  $T-Q$  relations (3.10), (3.11), with  $Q_1(u)$  and  $Q_2(u)$  given by (3.14) and  $h^{(2)}(u)$  given by (3.13). The  $h^{(1)}(u)$  is given by (3.18) and (3.19) respectively, for the two cases considered. The zeros  $\{u_j^{(1)}, u_j^{(2)}\}$  of  $Q_1(u)$  and  $Q_2(u)$  are indeed the solutions of the Bethe-ansatz-type equations, (3.20) and (3.21). These equations reproduce results in [46, 47] for  $s = \frac{1}{2}$ . In the following section, we shall use these results (specifically  $\tilde{\Lambda}^{(\frac{1}{2},s)}(u)$  which represents the eigenvalues of the rescaled “fundamental” transfer matrix given by (2.16)) to derive expressions for energy eigenvalues for the case  $s = 1$ .

## 4 Energy eigenvalues and Bethe roots

In this section, we provide numerical evidence for the completeness of the Bethe-ansatz-type solutions derived in Sec. 3 using the case  $s = 1$  as an example. We derive an expression for the energy eigenvalues for the open spin-1 XXZ quantum spin chain and compute the complete energy levels for this case from the Bethe roots given by (3.20) and (3.21). We do this for both cases, I and II.

### 4.1 Open spin-1 XXZ quantum spin chain: The Hamiltonian

In this section, we review the integrable Hamiltonian for the open spin-1 XXZ quantum spin chain (adopting notations used in [42]). The Hamiltonian is given by

$$\mathcal{H} = \sum_{n=1}^{N-1} H_{n,n+1} + H_b, \quad (4.1)$$

where  $H_{n,n+1}$  represents the bulk terms. Explicitly, these terms are given by [83],

$$\begin{aligned} H_{n,n+1} &= \sigma_n - (\sigma_n)^2 + 2 \operatorname{sh}^2 \eta \left[ \sigma_n^z + (S_n^z)^2 + (S_{n+1}^z)^2 - (\sigma_n^z)^2 \right] \\ &- 4 \operatorname{sh}^2 \left( \frac{\eta}{2} \right) (\sigma_n^\perp \sigma_n^z + \sigma_n^z \sigma_n^\perp), \end{aligned} \quad (4.2)$$

where

$$\sigma_n = \vec{S}_n \cdot \vec{S}_{n+1}, \quad \sigma_n^\perp = S_n^x S_{n+1}^x + S_n^y S_{n+1}^y, \quad \sigma_n^z = S_n^z S_{n+1}^z, \quad (4.3)$$

and  $\vec{S}$  are the  $su(2)$  spin-1 generators.  $H_b$  represents the boundary terms with the following form (see e.g., [42, 84])

$$\begin{aligned} H_b = & a_1(S_1^z)^2 + a_2S_1^z + a_3(S_1^+)^2 + a_4(S_1^-)^2 + a_5S_1^+ S_1^z + a_6S_1^z S_1^- \\ & + a_7S_1^z S_1^+ + a_8S_1^- S_1^z + (a_j \leftrightarrow b_j \text{ and } 1 \leftrightarrow N), \end{aligned} \quad (4.4)$$

where  $S^\pm = S^x \pm iS^y$ . The coefficients  $\{a_i\}$  of the boundary terms at site 1 are functions of the boundary parameters  $(\alpha_-, \beta_-, \theta_-)$  and the bulk anisotropy parameter  $\eta$ . They are given by,

$$\begin{aligned} a_1 &= \frac{1}{4}a_0 (\text{ch } 2\alpha_- - \text{ch } 2\beta_- + \text{ch } \eta) \text{sh } 2\eta \text{sh } \eta, \\ a_2 &= \frac{1}{4}a_0 \text{sh } 2\alpha_- \text{sh } 2\beta_- \text{sh } 2\eta, \\ a_3 &= -\frac{1}{8}a_0 e^{2\theta_-} \text{sh } 2\eta \text{sh } \eta, \\ a_4 &= -\frac{1}{8}a_0 e^{-2\theta_-} \text{sh } 2\eta \text{sh } \eta, \\ a_5 &= a_0 e^{\theta_-} \left( \text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} + \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \\ a_6 &= a_0 e^{-\theta_-} \left( \text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} + \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \\ a_7 &= -a_0 e^{\theta_-} \left( \text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} - \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \\ a_8 &= -a_0 e^{-\theta_-} \left( \text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} - \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \end{aligned} \quad (4.5)$$

where

$$a_0 = \left[ \text{sh}(\alpha_- - \frac{\eta}{2}) \text{sh}(\alpha_- + \frac{\eta}{2}) \text{ch}(\beta_- - \frac{\eta}{2}) \text{ch}(\beta_- + \frac{\eta}{2}) \right]^{-1}. \quad (4.6)$$

Similarly, the coefficients  $\{b_i\}$  of the boundary terms at site  $N$  which are functions of the boundary parameters  $(\alpha_+, \beta_+, \theta_+)$  and  $\eta$ , are given by the following correspondence,

$$b_i = a_i \Big|_{\alpha_- \rightarrow \alpha_+, \beta_- \rightarrow -\beta_+, \theta_- \rightarrow \theta_+}. \quad (4.7)$$

The Hamiltonian  $\mathcal{H}$  (4.1), according to [4], is related to the first derivative of the spin-1 transfer matrix, namely  $t^{(1,1)}(u)$ , which one constructs from  $t^{(\frac{1}{2},1)}(u)$  by using the fusion hierarchy formula (2.13),

$$t^{(1,1)}(u) = t^{(\frac{1}{2},1)}(u - \frac{\eta}{2}) t^{(\frac{1}{2},1)}(u + \frac{\eta}{2}) - \delta^{(1)}(u - \frac{\eta}{2}), \quad (4.8)$$

where  $\delta^{(1)}(u)$  is given by (2.14)-(2.15) with  $s = 1$ . Following [42], we work with the rescaled transfer matrix given by

$$\tilde{t}^{(1,1) \text{ } gt}(u) = \frac{\text{sh}(2u) \text{sh}(2u + 2\eta)}{[\text{sh } u \text{sh}(u + \eta)]^{2N}} t^{(1,1) \text{ } gt}(u), \quad (4.9)$$

where  $t^{(1,1) \text{ } gt}(u)$  is the transfer matrix constructed from “gauge”-transformed  $R^{(1,1)}(u)$  and  $K^{\mp(1)}(u)$  matrices<sup>3</sup>. We note here that the rescaled transfer matrix does not vanish at  $u = 0$ .

The Hamiltonian  $\mathcal{H}$  (4.1), can now be expressed in terms of the first derivative of  $\tilde{t}^{(1,1) \text{ } gt}(u)$ ,

$$\mathcal{H} = c_1^{(1)} \frac{d}{du} \tilde{t}^{(1,1) \text{ } gt}(u) \Big|_{u=0} + c_2^{(1)} \mathbb{I}, \quad (4.10)$$

where

$$\begin{aligned} c_1^{(1)} &= \text{ch } \eta \left\{ 16 [\text{sh } 2\eta \text{sh } \eta]^{2N} \text{sh } 3\eta \text{sh}(\alpha_- - \frac{\eta}{2}) \text{sh}(\alpha_- + \frac{\eta}{2}) \text{ch}(\beta_- - \frac{\eta}{2}) \text{ch}(\beta_- + \frac{\eta}{2}) \right. \\ &\quad \times \left. \text{sh}(\alpha_+ - \frac{\eta}{2}) \text{sh}(\alpha_+ + \frac{\eta}{2}) \text{ch}(\beta_+ - \frac{\eta}{2}) \text{ch}(\beta_+ + \frac{\eta}{2}) \right\}^{-1} \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} c_2^{(1)} &= -\frac{a_0}{4} b \text{ch } \eta - (N-1)(4 + \text{ch } 2\eta) + 2N \text{ch}^2 \eta \\ &\quad - \frac{\text{sh } \eta}{2d} \left\{ -2 \text{ch } 2\alpha_+ \left( \text{ch } \eta (3 + 7 \text{ch } 2\eta + \text{ch } 4\eta) + \text{ch } 2\beta_+ (4 + 5 \text{ch } 2\eta + 2 \text{ch } 4\eta) \right) \right. \\ &\quad + 2 \text{ch } \eta \left( \text{ch } 2\beta_+ (3 + 7 \text{ch } 2\eta + \text{ch } 4\eta) + \text{ch } \eta (5 + 3 \text{ch } 2\eta + 3 \text{ch } 4\eta) \right) \Big\} \\ &\quad - \frac{\text{sh } 2\eta}{2d} \left\{ \text{ch } 2\beta_+ (2 + 4 \text{ch } \eta \text{ch } 3\eta) + \text{ch } \eta (5 \text{ch } 2\eta + \text{ch } 4\eta) - 2 \text{ch } 2\alpha_+ \left( 1 + \text{ch } 2\eta \right. \right. \\ &\quad \left. \left. + \text{ch } 2\beta_+ (\text{ch } \eta + 2 \text{ch } 3\eta) + \text{ch } 4\eta \right) \right\}. \end{aligned} \quad (4.12)$$

In (4.12),  $b$  and  $d$  are given by

$$b = 2(-\text{ch } 2\beta_- - \text{ch}^3 \eta + \text{ch } 2\alpha_- (1 + \text{ch } 2\beta_- \text{ch } \eta)) \quad (4.13)$$

and

$$d = -4 \text{sh } 3\eta \text{sh}(\alpha_+ + \frac{\eta}{2}) \text{sh}(\alpha_+ - \frac{\eta}{2}) \text{ch}(\beta_+ + \frac{\eta}{2}) \text{ch}(\beta_+ - \frac{\eta}{2}). \quad (4.14)$$

---

<sup>3</sup>Such a transformation results in a more symmetric form of these matrices. For a detailed discussion on this, refer to Sec. 4 of [42].

## 4.2 Open spin-1 XXZ quantum spin chain: Energy eigenvalues

Next, we proceed to the eigenvalues of the Hamiltonian (4.10). Note that (4.10) implies the following result for the corresponding eigenvalues,

$$E = c_1^{(1)} \frac{d}{du} \tilde{\Lambda}^{(1,1)}(u) \Big|_{u=0} + c_2^{(1)}, \quad (4.15)$$

where  $\tilde{\Lambda}^{(1,1)}(u)$  represents the transfer matrix eigenvalues which assume the following form after using (2.16), (4.8) and (4.9),

$$\begin{aligned} \tilde{\Lambda}^{(1,1)}(u) &= \frac{\text{sh}(2u) \text{sh}(2u+2\eta)}{[\text{sh } u \text{sh}(u+\eta)]^{2N}} \left\{ [g^{(\frac{1}{2},s)}(u-\frac{\eta}{2}) g^{(\frac{1}{2},s)}(u+\frac{\eta}{2})]^{2N} \tilde{\Lambda}^{(\frac{1}{2},1)}(u-\frac{\eta}{2}) \tilde{\Lambda}^{(\frac{1}{2},1)}(u+\frac{\eta}{2}) \right. \\ &\quad \left. - \delta^{(1)}(u-\frac{\eta}{2}) \right\}, \end{aligned} \quad (4.16)$$

where  $\delta^{(1)}(u)$  is given by (2.14)-(2.15) with  $s = 1$ . Furthermore, we have also used the fact that  $\Lambda^{(1,1)}(u) = \Lambda^{(1,1)}(u)$ . Finally, from (3.11), (4.8) and (4.9), we obtain the energy in terms of Bethe roots  $\{u_k^{(j)}\}$  for cases I and II. Below, we present the analytic forms of the energy eigenvalues for these two cases:

Case I.  $\beta_-$  and  $\beta_+$  arbitrary while setting  $\alpha_{\pm} = 0$ ,  $\theta_- = \theta_+ = \theta = \text{arbitrary}$  :

$$\begin{aligned} E &= \text{sh}^2(2\eta) \sum_{k=1}^{M_1} \frac{1}{\text{sh}(u_k^{(1)} + \frac{3\eta}{2}) \text{sh}(u_k^{(1)} - \frac{\eta}{2})} + 2 \text{sh } 2\eta [(N+1) \text{cth } \eta - \text{cth } \frac{\eta}{2}] \\ &\quad + c_1^{(1)} C'(0) + c_2^{(1)} \end{aligned} \quad (4.17)$$

Case II.  $\alpha_-$  and  $\alpha_+$  arbitrary while setting  $\beta_{\pm} = 0$ ,  $\theta_- = \theta_+ = \theta = \text{arbitrary}$  :

$$\begin{aligned} E &= \text{sh}^2(2\eta) \sum_{k=1}^{M_1} \frac{1}{\text{sh}(u_k^{(1)} + \frac{3\eta}{2}) \text{sh}(u_k^{(1)} - \frac{\eta}{2})} + 2 \text{sh } 2\eta [(N+1) \text{cth } \eta - \text{th } \frac{\eta}{2}] \\ &\quad + c_1^{(1)} C'(0) + c_2^{(1)} \end{aligned} \quad (4.18)$$

where in (4.17) and (4.18),

$$C(u) = - \frac{\text{sh } 2u \text{sh}(2u+2\eta)}{[\text{sh } u \text{sh}(u+\eta)]^{2N}} \delta^{(1)}(u - \frac{\eta}{2}). \quad (4.19)$$

We recall that  $M_1 = N + \frac{1}{2}(p+1)$  and  $M_2 = M_1 - 1$ . Note that in (4.17) and (4.18), the contribution to  $E$  comes only from  $\{u_k^{(1)}\}$  and not from both  $\{u_k^{(1)}\}$  and  $\{u_k^{(2)}\}$  as one may initially expect. Perhaps, if one uses (3.10) instead of (3.11) in the derivation of  $E$ , an equivalent expression involving only  $\{u_k^{(2)}\}$  or both  $\{u_k^{(1)}\}$  and  $\{u_k^{(2)}\}$  may result. This however will not affect the numerical value of the energy eigenvalues tabulated in the next section.

### 4.3 Numerical results

In this section, we tabulate the energies computed using (4.17) and (4.18) for some values of  $N, p$  (therefore  $\eta$ ) and the boundary parameters  $\{\alpha_{\pm}, \beta_{\pm}, \theta_{\pm}\}$  with the Bethe roots,  $\{u_k^{(1)}\}$ , in Tables 1 and 2 for cases I and II respectively. These Bethe roots are obtained using McCoy's method [81, 82]. These numerical results demonstrate the completeness of the Bethe-ansatz-type equations, (3.20) and (3.21) for the case  $s = 1$ . We checked these solutions for chains of length up to  $N = 4$  for  $p = 3, 5$  and  $7$  with boundary parameters  $\beta_+ = 0.695, \beta_- = 0.774$ . We remark that these solutions reproduce the  $s = \frac{1}{2}$  case given in [46, 47]. The  $T - Q$  relations (3.10) and (3.11) are also numerically verified for  $s = \frac{3}{2}$  case for some selected values of  $N, p$  and boundary parameters. We acknowledge that these analysis provide some numerical support for the completeness of the Bethe-ansatz-type equations derived, (3.20) and (3.21), and not a complete rigorous proof. We have also verified that the energies given in Tables 1 and 2 coincide with those obtained from direct diagonalization of (4.1).

## 5 Discussion

By using a method that relies on certain functional relations that the “fundamental” transfer matrices,  $t^{(\frac{1}{2}, s)}(u)$ , obey at roots of unity and the truncation of fusion hierarchy, we set up a generalized form of the  $T - Q$  relation, (3.10) and (3.11), for the open spin- $s$  XXZ quantum spin chain with nondiagonal boundary terms. From these relations, we have determined Bethe-ansatz-type solutions of the model, (3.20) and (3.21). These solutions hold only for  $\eta = \frac{i\pi}{2}, \frac{i\pi}{4}, \dots$ . The solutions found here hold for arbitrary values of boundary parameters (at most two). These solutions have been checked for chains of length up to  $N = 4$  for  $p = 3, 5$  and  $7$  with boundary parameters  $\beta_+ = 0.695, \beta_- = 0.774$ . We verified that they indeed produce all the  $(2s + 1)^N$  transfer matrix eigenvalues for  $s = \frac{1}{2}, 1$  and  $\frac{3}{2}$ . Moreover, we also presented numerical evidence for the completeness of the Bethe-ansatz-type solutions found (using  $s = 1$  as examples) in Tables 1 and 2. The numerical support for the completeness of the solutions presented here (using  $s = 1$  case as examples) together with the results presented for spin-1/2 case in [46, 47] and the fusion hierarchy (2.13) which is used in the construction of higher spin- $s$  transfer matrices could perhaps possibly enable one to develop a more formal rigorous proof for the completeness of the solutions found here. It would be interesting to pursue this in the future.

In addition, a number of problems remain that are worth investigating. Perhaps one could carry out a more thorough treatment and analysis of the functional equation to yield

the exact form of the  $Q_i(u)$  functions, thus avoiding the need for an ansatz such as (3.14). Another interesting problem is to see the relation of  $s = 1$  case to the supersymmetric sine-Gordon (SSG) model, along the lines of [58] and [59], but now for spin-1 chain with nondiagonal boundary terms described by the generalized  $T - Q$  relations instead of the conventional  $T - Q$  relation. One could also try to generalize the solutions presented in [48] for the spin-1/2 case, where all six boundary parameters are completely arbitrary, to any spin  $s$ , and analyze the  $s = 1$  case for this general solution in relation to the SSG model. In this regard, one can study the continuum limit of their Nonlinear Integral Equations (NLIEs), thus investigating the infrared (IR) and ultraviolet (UV) limits of the NLIEs. One could also investigate the boundary bound states of SSG models corresponding to all these cases such as reported recently in [85]. We hope to be able to address some of these issues in future publications.

## Acknowledgments

We thank the referees for their constructive and crucial comments and suggestions which greatly helped us in revising and improving the paper.

## References

- [1] M. Gaudin, “Boundary Energy of a Bose Gas in One Dimension,” *Phys. Rev.* **A4**, 386 (1971).
- [2] M. Gaudin, *La fonction d’onde de Bethe* (Masson, 1983).
- [3] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, “Surface exponents of the quantum XXZ, Ashkin-Teller and Potts models,” *J. Phys.* **A20**, 6397 (1987).
- [4] E.K. Sklyanin, “Boundary conditions for integrable quantum systems,” *J. Phys.* **A21**, 2375 (1988).
- [5] L. Mezincescu, R.I. Nepomechie and V. Rittenberg, “Bethe Ansatz solution of the Fateev-Zamolodchikov quantum spin chain with boundary terms,” *Phys. Lett.* **A147**, 70 (1990).
- [6] E.C. Fireman, A. Lima-Santos and W. Utiel, “Bethe Ansatz solution for quantum spin-1 chains with boundary terms,” *Nucl. Phys.* **B626**, 435 (2002) [[nlin/0110048](#)].



- [7] W. Galleas and M.J. Martins, “Solution of the SU(N) Vertex Model with Non-Diagonal Open Boundaries,” *Phys. Lett.* **A335**, 167 (2005) [[nlin.SI/0407027](#)].
- [8] C.S. Melo, G.A.P. Ribeiro and M.J. Martins, “Bethe ansatz for the XXX-S chain with non-diagonal open boundaries,” *Nucl. Phys.* **B711**, 565 (2005) [[nlin.SI/0411038](#)].
- [9] A. Doikou and A. Babichenko, “Principal chiral model scattering and the alternating quantum spin chain,” *Phys. Lett* **B515**, 220 (2001) [[hep-th/0105033](#)].
- [10] A. Doikou, “The XXX spin s quantum chain and the alternating  $s^1$ ,  $s^2$  chain with boundaries ,” *Nucl. Phys.* **B634**, 591 (2002) [[hep-th/0201008](#)].
- [11] A. Doikou and P.P. Martin, “ On quantum group symmetry and Bethe ansatz for the asymmetric twin spin chain with integrable boundary,” *J. Stat. Mech.* **P06004** (2006) [[hep-th/0503019](#)].
- [12] A. Doikou, “The Open XXZ and associated models at q root of unity,” *J. Stat. Mech.* **P09010** (2006) [[hep-th/0603112](#)].
- [13] J. de Gier and P. Pyatov, “Bethe Ansatz for the Temperley-Lieb loop model with open boundaries,” *J. Stat. Mech.* **P03002** (2004) [[hep-th/0312235](#)].
- [14] A. Nichols, V. Rittenberg and J. de Gier, “One-boundary Temperley-Lieb algebras in the XXZ and loop models,” *J. Stat. Mech.* **P03003** (2005) [[cond-mat/0411512](#)].
- [15] J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg, “Magic in the spectra of the XXZ quantum chain with boundaries at  $\Delta = 0$  and  $\Delta = -1/2$ ,” *Nucl. Phys.* **B729**, 387 (2005) [[hep-th/0505062](#)].
- [16] J. de Gier and F.H.L. Essler, “Bethe Ansatz Solution of the Asymmetric Exclusion Process with Open Boundaries,” *Phys. Rev. Lett.* **95**, 240601 (2005) [[cond-mat/0508707](#)].
- [17] J. de Gier and F.H.L. Essler, “Exact spectral gaps of the asymmetric exclusion process with open boundaries,” *J. Stat. Mech.* **P12011** (2006) [[cond-mat/0609645](#)].
- [18] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, “General boundary conditions for the  $sl(N)$  and  $sl(M|N)$  open spin chains,” *J. Stat. Mech.* **P08005**, (2004) [[math-ph/0406021](#)].
- [19] W.-L. Yang, Y.-Z. Zhang and M. Gould, “Exact solution of the XXZ Gaudin model with generic open boundaries,” *Nucl. Phys.* **B698**, 503 (2004) [[hep-th/0411048](#)].
- [20] W.-L. Yang and Y.-Z. Zhang, “Exact solution of the  $A_{n-1}^{(1)}$  trigonometric vertex model with non-diagonal open boundaries,” *JHEP* **01**, 021 (2005) [[hep-th/0411190](#)].

- [21] W.-L. Yang, Y.-Z. Zhang and R. Sasaki, “ $A_{n-1}$  Gaudin model with open boundaries,” *Nucl. Phys.* **B729**, 594 (2005) [[hep-th/0507148](#)].
- [22] P. Baseilhac and K. Koizumi, “A deformed analogue of Onsager’s symmetry in the XXZ open spin chain,” *J. Stat. Mech.* **P10005** (2005) [[hep-th/0507053](#)].
- [23] P. Baseilhac, “The  $q$ -deformed analogue of the Onsager algebra: beyond the Bethe ansatz approach,” *Nucl. Phys.* **B754**, 309 (2006) [[math-ph/0604036](#)].
- [24] P. Baseilhac and K. Koizumi, “Exact spectrum of the XXZ open spin chain from the  $q$ -Onsager algebra representation theory,” *J. Stat. Mech.* **P09006** (2007) [[hep-th/0703106](#)].
- [25] P. Baseilhac, “New results in the XXZ open spin chain,” [[hep-th/0712.0452](#)].
- [26] A. Nichols, “The Temperley-Lieb algebra and its generalizations in the Potts and XXZ models,” *J. Stat. Mech.* **P01003** (2006) [[hep-th/0509069](#)].
- [27] A. Nichols, “Structure of the two-boundary XXZ model with non-diagonal boundary terms,” *J. Stat. Mech.* **L02004** (2006) [[hep-th/0512273](#)].
- [28] Z. Bajnok, “Equivalences between spin models induced by defects,” *J. Stat. Mech.* **P06010** (2006) [[hep-th/0601107](#)].
- [29] W. Galleas, “Functional relations from the Yang-Baxter algebra: Eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions,” *Nucl. Phys.* **B790**, 524 (2008) [[nlin.SI/0708.0009](#)].
- [30] N. Crampe, E. Ragoucy and D. Simon, “Eigenvectors of open XXZ and ASEP models for a class of non-diagonal boundary conditions,” *J. Stat. Mech.* **P11038** (2010) [[cond-mat/1009.4119](#)].
- [31] N. Crampe, E. Ragoucy and D. Simon, “Matrix Coordinate Bethe Ansatz: Applications to XXZ and ASEP models,” *J. Phys.* **A44**, 405003 (2011) [[cond-mat/1106.4712](#)].
- [32] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, “Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields,” [[cond-mat/0212163](#)].
- [33] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, “Exact solution of XXZ spin chain with unparallel boundary fields,” *Nucl. Phys.* **B663**, 487 (2003).
- [34] R.I. Nepomechie, “Solving the open XXZ spin chain with nondiagonal boundary terms at roots of unity,” *Nucl. Phys.* **B622**, 615 (2002); Addendum, *Nucl. Phys.* **B631**, 519 (2002) [[hep-th/0110116](#)].

- [35] R.I. Nepomechie, “Functional relations and Bethe Ansatz for the XXZ chain,” *J. Stat. Phys.* **111**, 1363 (2003) [[hep-th/0211001](#)].
- [36] R.I. Nepomechie, “Bethe Ansatz solution of the open XXZ chain with nondiagonal boundary terms,” *J. Phys.* **A37**, 433 (2004) [[hep-th/0304092](#)].
- [37] R.I. Nepomechie and F. Ravanini, “Completeness of the Bethe Ansatz solution of the open XXZ chain with nondiagonal boundary terms,” *J. Phys.* **A36**, 11391 (2003); Addendum, *J. Phys.* **A37**, 1945 (2004) [[hep-th/0307095](#)].
- [38] W.-L. Yang, R.I. Nepomechie and Y.-Z. Zhang, “Q-operator and T-Q relation from the fusion hierarchy,” *Phys. Lett.* **B633**, 664 (2006) [[hep-th/0511134](#)].
- [39] W.-L. Yang and Y.-Z. Zhang, “On the second reference state and complete eigenstates of the open XXZ chain,” *JHEP* **04**, 044 (2007) [[hep-th/0703222](#)].
- [40] A. Doikou, “Fused integrable lattice models with quantum impurities and open boundaries,” *Nucl. Phys.* **B668**, 447 (2003) [[hep-th/0303205](#)].
- [41] A. Doikou, “A note on the boundary spin  $s$  XXZ chain,” *Phys. Lett.* **A366**, 556 (2007) [[hep-th/0612268](#)].
- [42] L. Frappat, R.I. Nepomechie and E. Ragoucy, “Complete Bethe ansatz solution of the open spin- $s$  XXZ chain with general integrable boundary terms,” *J. Stat. Mech.* **P09009** (2007) [[math-ph/0707.0653](#)].
- [43] R. J. Baxter, “Exactly Solved Models in Statistical Mechanics,” *New York: Academic* (1982)
- [44] W.-L. Yang and Y.-Z. Zhang, “ $T$ - $Q$  relation and exact solution for the XYZ chain with general nondiagonal boundary terms,” *Nucl. Phys.* **B744**, 312 (2006) [[hep-th/0512154](#)].
- [45] R.Murgan, “Bethe ansatz of the open spin- $s$  XXZ chain with nondiagonal boundary terms,” *JHEP* **04**, 076 (2009) [[hep-th/0901.3558](#)].
- [46] R. Murgan and R.I. Nepomechie, “Generalized  $T - Q$  relations and the open XXZ chain,” *J. Stat. Mech.* **P08002** (2005) [[hep-th/0507139](#)].
- [47] R. Murgan, R.I. Nepomechie and C. Shi, “Boundary energy of the open XXZ chain from new exact solutions,” *Ann. Henri Poincare*, **7**, 1429 (2006) [[hep-th/0512058](#)].
- [48] R. Murgan, R.I. Nepomechie and C. Shi, “Exact solution of the open XXZ chain with general integrable boundary terms at roots of unity,” *J. Stat. Mech* **P08006** (2006) [[hep-th/0605223](#)].

- [49] P. Di Vecchia and S. Ferrara, “Classical solutions in two-dimensional supersymmetric field theories,” *Nucl. Phys.* **B130**, 93 (1977).
- [50] J. Hruby, “On the supersymmetric sine-Gordon model and a two-dimensional ‘bag’” *Nucl. Phys.* **B131**, 275 (1977).
- [51] S. Ferrara, L. Girardello and S. Sciuto, “An infinite set of conservation laws of the supersymmetric sine-Gordon theory,” *Phys. Lett.* **B76**, 303 (1978).
- [52] R. Shankar and E. Witten, “The S matrix of the supersymmetric nonlinear sigma model,” *Phys. Rev.* **D17**, 2134 (1978).
- [53] C. Ahn, D. Bernard and A. LeClair, “Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory,” *Nucl. Phys.* **B346**, 409 (1990).
- [54] C. Ahn, “Complete S matrices of supersymmetric sine-Gordon theory and perturbed superconformal minimal model,” *Nucl. Phys.* **B354**, 57 (1991).
- [55] T. Inami, S. Odake and Y.-Z. Zhang, “Supersymmetric extension of the sine-Gordon theory with integrable boundary interactions,” *Phys. Lett.* **B359**, 118 (1995) [[hep-th/9506157](#)].
- [56] R.I. Nepomechie, “The boundary supersymmetric sine-Gordon model revisited,” *Phys. Lett.* **B509**, 183 (2001) [[hep-th/0103029](#)].
- [57] Z. Bajnok, L. Palla and G. Takács, “Spectrum of boundary states in  $N = 1$  SUSY sine-Gordon theory,” *Nucl. Phys.* **B644**, 509 (2002) [[hep-th/0207099](#)].
- [58] C. Ahn, R.I. Nepomechie and J. Suzuki, “Finite size effects in the spin-1 XXZ and supersymmetric sine-Gordon models with Dirichlet boundary conditions,” *Nucl. Phys.* **B767**, 250 (2007) [[hep-th/0611136](#)].
- [59] R. Murgan, “A note on the IR limit of the NLIEs of boundary supersymmetric sine-Gordon model,” *JHEP* **09**, 059 (2011) [[hep-th/1107.1928](#)].
- [60] P.P. Kulish and E.K. Sklyanin, “Quantum spectral transform method, recent developments,” *Lecture Notes in Physics*, Vol. 151, 61 (Springer, 1982).
- [61] P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, “Yang-Baxter equation and representation theory. I,” *Lett. Math. Phys.* **5**, 393 (1981).
- [62] P.P. Kulish and N.Yu. Reshetikhin, “Quantum linear problem for the sine-Gordon equation and higher representation,” *J. Sov. Math.* **23**, 2435 (1983).

- [63] A.N. Kirillov and N.Yu. Reshetikhin, “Exact solution of the Heisenberg XXZ model of spin  $s$ ,” *J. Sov. Math.* **35**, 2627 (1986).
- [64] A.N. Kirillov and N.Yu. Reshetikhin, “Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum,” *J. Phys.* **A20**, 1565 (1987).
- [65] L. Mezincescu and R.I. Nepomechie, “Fusion procedure for open chains,” *J. Phys.* **A25**, 2533 (1992).
- [66] Y.-K. Zhou, “Row transfer matrix functional relations for Baxter’s eight-vertex and six-vertex models with open boundaries via more general reflection matrices,” *Nucl. Phys.* **B458**, 504 (1996) [[hep-th/9510095](#)].
- [67] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, “Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz,” *Commun. Math. Phys.* **177**, 381 (1996) [[hep-th/9412229](#)].
- [68] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, “Integrable structure of conformal field theory III. The Yang-Baxter relation,” *Commun. Math. Phys.* **200**, 297 (1999) [[hep-th/9805008](#)].
- [69] A. Kuniba, K. Sakai and J. Suzuki, “Continued fraction TBA and functional relations in XXZ model at root of unity,” *Nucl. Phys.* **B525**[FS], 597 (1998) [[math/9803056](#)].
- [70] V.V. Bazhanov and N.Yu. Reshetikhin, “Critical RSOS Models And Conformal Field Theory,” *Int. J. Mod. Phys.* **A4**, 115 (1989).
- [71] M. Karowski, “On the bound state problem in (1+1)-dimensional field theories” *Nucl. Phys.* **B153**, 244 (1979).
- [72] H.M. Babujian, “Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model,” *Nucl. Phys.* **B215**, 317 (1983).
- [73] L.A. Takhtajan, “The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins,” *Phys. Lett.* **87A**, 479 (1982).
- [74] K. Sogo, “Ground state and low-lying excitations in the Heisenberg XXZ chain of arbitrary spin  $S$ ,” *Phys. Lett.* **A104**, 51 (1984).
- [75] H.M. Babujian and A.M. Tsvelick, “Heisenberg magnet with an arbitrary spin and anisotropic chiral field,” *Nucl. Phys.* **B265** [FS15], 24 (1986).

- [76] H.J. de Vega and A. González-Ruiz, “Boundary K-matrices for the six vertex and the  $n(2n - 1)$   $A_{n-1}$  vertex models,” *J. Phys.* **A26**, L519 (1993) [[hep-th/9211114](#)].
- [77] S. Ghoshal and A.B. Zamolodchikov, “Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory,” *Int. J. Mod. Phys.* **A9**, 3841 (1994) [[hep-th/9306002](#)].
- [78] R. J. Baxter, “Partition function of the eight-vertex lattice model,” *Ann. Phys. (NY)* **70**, 193 (1972) [*Ann. Phys. (NY)* **281**, 187 (2000)].
- [79] R. J. Baxter, “Asymptotically degenerate maximum eigenvalues of the eight-vertex model transfer matrix and interfacial tension,” *J. Stat. Phys.* **8**, 25 (1973).
- [80] I.V. Cherednik, “Factorizing particles on a half line and root systems,” *Theor. Math. Phys.* **61**, 977 (1984).
- [81] S. Dasmahapatra, R. Kedem and B.M. McCoy, “Spectrum and completeness of the three state superintegrable chiral Potts model” *Nucl. Phys.* **B396**, 506 (1993) [[hep-th/9204003](#)].
- [82] K. Fabricius and B.M. McCoy, “Bethe’s equation is incomplete for the XXZ model at roots of unity” *J. Stat. Phys.* **103**, 647 (2001) [[cond-mat/0009279](#)].
- [83] A.B. Zamolodchikov and V.A. Fateev, “Model factorized S matrix and an integrable Heisenberg chain with spin 1,” *Sov. J. Nucl. Phys.* **32**, 298 (1980).
- [84] T. Inami, S. Odake and Y.-Z. Zhang, “Reflection K matrices of the 19 vertex model and XXZ spin 1 chain with general boundary terms,” *Nucl. Phys.* **B470**, 419 (1996) [[hep-th/9601049](#)].
- [85] C. Matsui, “Boundary bound states in the SUSY sine-Gordon model with Dirichlet boundary conditions,” [[hep-th/1205.0912](#)].

$E$	Bethe roots, $\{u_k^{(1)}\}$
-5.6483	$0.426847 + 2.19193 i, 0.719676 + 1.1781 i, 0.109151 i,$ $0.426847 + 0.164266 i$
-4.67715	$0.106242 + 2.28424 i, 0.379199 + 1.1781 i, 1.05101 + 1.1781 i,$ $0.106242 + 0.071957 i$
-2.75841	$0.387014 + 2.748893 i, 1.277532 i, 0.932369 + 1.1781 i,$ $0.0609966 i$
-1.98286	$0.185547 + 2.748893 i, 1.701637 i, 0.915819 + 1.1781 i,$ $0.138044 i$
-1.54571	$0.171807 + 3.046499 i, 0.171807 + 2.451287 i, 1.566925 i,$ $0.916569 + 1.1781 i$
-0.489791	$0.781754 + 1.921787i, 1.599981 i, 0.0312436 i,$ $0.781754 + 0.434407 i$
-0.392189	$3.109568 i, 0.779636 + 1.920991 i, 1.554992 i,$ $0.779636 + 0.435203 i$
0.572634	$0.810472 i, 0.624212 + 1.1781 i, 0.010646 i,$ $1.227343 + 1.1781 i$
0.808501	$3.130312 i, 0.791507 i, 0.618753 + 1.1781 i,$ $1.221033 + 1.1781 i$

Table 1: The 9 energies and corresponding Bethe roots for  $N = 2, s = 1, p = 3, \eta = i\pi/4, \alpha_- = 0, \beta_- = 0.767, \theta_- = 0.573, \alpha_+ = 0, \beta_+ = 0.598, \theta_+ = 0.573$

$E$	Bethe roots, $\{u_k^{(1)}\}$
-6.07709	$0.0471453 + 3.1415 i, 0.0471453 + 2.61809 i, 1.74867 i, 0.74532 + 1.309 i,$ $0.48742 i$
-4.65604	$2.65564 i, 0.107433 + 2.35618 i, 0.321204 + 1.309 i, 0.557414 i,$ $0.107433 + 0.261819 i$
-4.3506	$0.00657235 + 3.07819 i, 0.00657235 + 2.6814 i, 2.07693 i, 0.12098 + 1.93837 i,$ $0.12098 + 0.679624 i$
-2.55991	$0.272597 + 3.13706 i, 0.272597 + 2.62253 i, 2.13098 i, 0.672718 + 1.309 i,$ $0.862768 i$
-1.63092	$0.326829 + 2.87979 i, 0.308315 + 2.35663 i, 2.13093 i, 0.890835 i,$ $0.308315 + 0.261367 i$
0.0925845	$0.248529 + 2.87979 i, 1.76311 i, 0.373083 + 1.309 i, 1.20497 + 1.309 i,$ $0.487 i$
0.0971716	$0.548694 + 2.59187 i, 2.13099 i, 0.518481 + 1.309 i, 0.856853 i,$ $0.548694 + 0.0261235 i$
1.6757	$0.70468 + 2.87979 i, 0.338436 + 1.309 i, 0.854426 i, 1.08306 + 1.309 i,$ $0.487 i$
2.99332	$1.7639 i, 0.273003 + 1.309 i, 0.720682 + 1.309 i, 0.487 i,$ $1.54847 + 1.309 i$

Table 2: The 9 energies and corresponding Bethe roots for  $N = 2, s = 1, p = 5, \eta = i\pi/6, \alpha_- = 0.854i, \beta_- = 0, \theta_- = 0.482, \alpha_+ = 0.487i, \beta_+ = 0, \theta_+ = 0.482$